
OPTIMAL Q-VALUE FUNCTIONS FOR DEC-POMDP

Shuo Liu
 Computer Science
 Northeastern University
 shuo.liu2@northeastern.edu

ABSTRACT

This article discusses the optimal Q-value function definition in Dec-POMDP.

1 NOTIONS

s^t	the state at t with problem horizon h
o^t	the joint observation of agents $o^t = \langle o_1^t, \dots, o_n^t \rangle$ at t
\mathcal{O}	the joint observation space
$\vec{\theta}^t$	the joint observation-action history until t , $\vec{\theta}^t = (o^0, a^0, \dots, o^t)$
$\vec{\Theta}^t$	the joint history space at t
$\vec{\Theta}_\pi^t$	the set of $\vec{\theta}^t$ consistent with policy π
δ^t	the decision rule (a temporal structure of policy) at t
$\delta^{t,*}$	the optimal decision rule at t following $\psi^{t-1,*}$
$\delta_\psi^{t,\otimes}$	the optimal decision rule at t following ψ^{t-1}
Δ^t	the decision rule space at t
ψ^t	the past joint policy until t , $\psi^t = \delta^{[0,t]}$
$\psi^{t,*}$	the optimal past joint policy until t , $\psi^{t,*} = \delta^{[0,t],*}$
$\psi^{t,\otimes}$	the past joint policy until t with non-optimal ψ^{t-1} and optimal $\delta_\psi^{t-1,\otimes}$
Ψ^t	the past joint policy space at t
ξ^t	the subsequent joint policy from t , $\xi^t = \delta^{[t,h]}$
$\xi^{t,*}$	the optimal subsequent joint policy from t , $\xi^{t,*} = \delta^{[t,h],*}$
$\xi_\psi^{t,\otimes}$	the optimal subsequent joint policy from t following non-optimal ψ^t
π	the joint pure policy $\pi = \delta^{[0,h]}$
π^*	the joint optimal pure policy $\pi^* = \delta^{[0,h],*}$
$R(\vec{\theta}^t, \psi^{t+1})$	the immediate reward function following ψ^{t+1}
$Q(\vec{\theta}^t, \psi^{t+1})$	the history-policy value function following ψ^{t+1}
$Q^*(\vec{\theta}^t, \psi^{t+1})$	the optimal history-policy value function following ψ^{t+1}
$Q^{\otimes}(\vec{\theta}^t, \psi^{t+1})$	the sequentially rational optimal history-policy value function following ψ^{t+1}

2 NORMATIVE OPTIMAL Q-VALUE FUNCTION

Definition 1. The optimal Q-value function Q^* in Dec-POMDP, the expected cumulative reward over time steps $[t, h)$ induced by optimal joint policy π^* , $\forall \bar{\theta}^t \in \bar{\Theta}_{\psi^{t,*}}^t, \forall \psi^{t+1} \in (\psi^{t,*}, \Delta^t)$, is defined as,

$$Q^*(\bar{\theta}^t, \psi^{t+1}) = \begin{cases} R(\bar{\theta}^t, \psi^{t+1}), & t = h - 1 \\ R(\bar{\theta}^t, \psi^{t+1}) + \sum_{o^{t+1} \in \mathcal{O}} P(o^{t+1} | \bar{\theta}^t, \psi^{t+1}) Q^*(\bar{\theta}^{t+1}, \pi^*(\bar{\theta}^{t+1})). & 0 \leq t < h - 1 \end{cases} \quad (1)$$

Here, $\pi^*(\bar{\theta}^{t+1}) \equiv \psi^{t+2,*}$ because of the consistent optimality of policy.

Proposition 1. In Dec-POMDP, deriving an optimal policy from the normative optimal history-policy value function defined in Equ. 1 is impractical (clarifying Sec. 4.3.3, Oliehoek et al. (2008)).

Proof. We check the optima in 2 steps. The independent and dependent variables are marked in red. To calculate the Pareto optima of Bayesian game at t ,

$$\delta^{t,*} = \operatorname{argmax}_{\delta^t} \sum_{\bar{\theta}^t \in \bar{\Theta}_{\psi^{t,*}}^t} P(\bar{\theta}^t | \psi^{t,*}) Q^*(\bar{\theta}^t, (\psi^{t,*}, \delta^t)), \quad (2)$$

note that calculating $\delta^{t,*}$ depends on $\psi^{t,*} = \delta^{[0,t],*}$ and $Q^*(\bar{\theta}^t, \cdot)$.

According to Definition. 1, the optimal Bellman equation can be written as,

$$Q^*(\bar{\theta}^t, \psi^{t+1}) = R(\bar{\theta}^t, \psi^{t+1}) + \sum_{o^{t+1} \in \mathcal{O}} P(o^{t+1} | \bar{\theta}^t, \psi^{t+1}) \max_{\delta^{t+1}} Q^*(\bar{\theta}^{t+1}, (\psi^{t+1,*}, \delta^{t+1})), \quad (3)$$

when $0 \leq t < h-1$. This indicates that $Q^*(\bar{\theta}^t, \cdot)$ depends on $\psi^{t+1,*}$.¹ Consequently, calculating $\delta^{t,*}$ inherently depends on $\delta^{[0,t],*}$ (includes itself), making it self-dependent and impractical to solve.² \square

3 SEQUENTIALLY RATIONAL OPTIMAL Q-VALUE FUNCTION

To make optimal Q-value in Dec-POMDP computable, Oliehoek et al. (2008) defined another form of Q-value function and eliminated the dependency on past optimality.

Definition 2. The sequentially rational optimal Q-value function Q^\circledast in Dec-POMDP, the expected cumulative reward over time steps $[t, h)$ induced by optimal subsequent joint policy $\xi_\psi^{t,\circledast}$, $\forall \bar{\theta}^t \in \bar{\Theta}_{\Psi^t}^t, \forall \psi^{t+1} \in \Psi^{t+1}$, is defined as,

$$Q^\circledast(\bar{\theta}^t, \psi^{t+1}) = \begin{cases} R(\bar{\theta}^t, \psi^{t+1}), & t = h - 1 \\ R(\bar{\theta}^t, \psi^{t+1}) + \sum_{o^{t+1} \in \mathcal{O}} P(o^{t+1} | \bar{\theta}^t, \psi^{t+1}) Q^\circledast(\bar{\theta}^{t+1}, \psi^{t+2,\circledast}), & 0 \leq t < h - 1 \end{cases} \quad (4)$$

where $\psi^{t+2,\circledast} = (\psi^{t+1}, \delta_\psi^{t+1,\circledast}), \forall \psi^{t+1} \in \Psi^{t+1}$.

Note that the only difference of Q^\circledast from Q^* is $\psi^{t+2,\circledast}$, consequently expanding Q^* 's candidates of history from $\bar{\theta}^t \in \bar{\Theta}_{\psi^{t,*}}^t$ to $\bar{\theta}^t \in \bar{\Theta}_{\Psi^t}^t$ and policy from $\psi^{t+1} \in (\psi^{t,*}, \Delta^t)$ to $\psi^{t+1} \in (\Psi^t, \Delta^t)$.

Beyond solving the problem of Proposition 1, another advantage of Q^\circledast is that it allows for the computation of optimal subsequent policy $\xi_\psi^{t,*}$ following any past policy ψ^t . This is beneficial in online applications where agents may occasionally deviate from the optimal policy.

¹The dependency of $P(o^{t+1} | \bar{\theta}^t, \psi^{t+1})$ is not a problem and can be solved just like how the stochasticity $P(s^{t+1} | s^t, a)$ tackled by double learning in Sec. 6.7, Sutton & Barto (2018).

²Single-agent (PO)MDP, where the belief states are acquirable, does not have such a problem because the Q-value function is not necessarily history-dependent, thanks to Markovian property.

4 OPEN QUESTIONS

- We have seen some advantages of defining the optimal Q-value function as Q^* , what are the downsides to defining it this way (e.g., high computational costs)?

REFERENCES

Frans A. Oliehoek, Matthijs T. J. Spaan, and Nikos Vlassis. Optimal and approximate Q-value functions for decentralized POMDPs. *Journal of Artificial Intelligence Research*, 32:289–353, 2008.

Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, 2nd edition, 2018.